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About Newton's law of gravity

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Probably nobody will doubt that Newton's law of gravity is the most perfect summary of all experiential facts about the movements within our planetary system. There is account and explanation of all motion processes in detail and also the few anomalies not yet explained do not necessarily indicate an imperfection of the law. If we have to consider the validity of Newton's law within our planetary system as one of the most reliable results of theoretical astronomy, we must not ignore the fact that up to now there are no observational facts that can guarantee or prove its universal validity. It will be a permissible conclusion by analogy if we assume that the law of attraction is independent of the position of the attracting masses in space, i.e. that it rules within the multiple star systems, the star clusters as well as in planetary systems. We will not doubt the validity of this assumption even if we have recognized that the observed double star motions are not very suitable to decide this question in a safe way.[1] But things are quite different if we ask ourselves the question whether Newton's law also accurately reflects the attraction of masses separated by immeasurable distances. Experience has shown that there is no direct point of reference here and, on the other hand, Newton's law is a purely empirical formula whose absolute accuracy we know nothing about. Therefore, the question is allowed whether we may extend Newton's law to immeasurably large spaces, or whether this procedure may lead to contradictions or difficulties. The following lines contain considerations to this effect.

For the sake of simplicity - because it is easy to see that this assumption does not overturn the following considerations - let us think of all bodies filling the universe as spheres whose density is arranged in concentric layers. The attraction of any world body to any point outside it is not changed if its mass is pulled apart into a concentric sphere of any size and only one concentrically arranged but otherwise arbitrary density of mass is maintained. It is necessary that the attracted point lies outside the larger sphere and that the total mass remains unchanged. If the large sphere has a finite diameter, however large, its mass density can be considered finite at all points. If the above process is carried out in a suitable manner for all the bodies of the universe, the attraction experienced by any point *A* will be equal to the attraction of a space filled everywhere with mass of finite density δ. The same space will surround all the bodies under consideration on the outside and on the inside to point *A* on all sides. The inner cavity contains no mass. Now the potential of this mass distribution on *A* is to be calculated. In the immediate vicinity of *A* lies the coordinate beginning *O*. It is also ρ and r the distance of a mass element d*m* from *A* and *O*, γ the angle *AOdm* and φ the angle which the plane *AOdm* forms with a fixed plane passing through *AO*. If the attraction constant is then set to 1, then

$$V = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \sin\gamma \, \mathrm{d}\gamma \int_{R_0}^{R_1} \frac{\delta r^2 \mathrm{d}r}{\rho}$$

R_0 and R_1 are the values of r, which define the inner and outer boundary of the space filled with mass. If we still set $AO=a$ and develop because $\frac{a}{r}$ is a small fraction under all circumstances, according to the potencies of that fraction, then

$$\frac{1}{\rho} = \frac{1}{r}\sum_{n=0}^{\infty} \frac{a^n}{r^n} P^n(\cos\gamma)$$

Where *Pn* is the famous Laplace-Legendre's spherical function. Herewith

$$V = \sum_{n=0}^{\infty} a^n \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\gamma \sin\gamma \, P^n(\cos\gamma) \int_{R_0}^{R_1} \mathrm{d}r \frac{\delta}{r^{n-1}}$$

And for $a=0$, the result is

$$V = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\gamma \sin\gamma \int_{R_0}^{R_1} \mathrm{d}r \cdot r \cdot \delta$$

$$X = \frac{\partial V}{\partial a} = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\gamma \sin\gamma \, P^1(\cos\gamma) \int_{R_0}^{R_1} \mathrm{d}r \cdot \delta$$

$$Z = \frac{\partial^2 V}{\partial a^2} = 2\int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \mathrm{d}\gamma \sin\gamma \, P^2(\cos\gamma) \int_{R_0}^{R_1} \frac{\mathrm{d}r}{r} \delta$$

$\frac{\partial V}{\partial a} = X$ is the acceleration which the attracted point experiences in direction *a* If one imagines a small extended mass to be exposed to the attraction, then $\frac{\partial^2 V}{\partial a^2}\Delta a$ the acceleration with which 2 points in the very small mutual distance Δa seem to move away from each other in this direction. It can be said that the mass experiences a strain = *Z* in the direction *a*.

[1] Compare this to my second pickup at ζ Cancri, S.

R_0 is always a finite quantity. If now the effect of a finite part of space is considered, then R_1 is also a finite quantity. The same applies to *V*, *X* and *Z*, and the conditions that *X* or *Z* become zero, for example, can easily be read from the equations. For the whole universe, R_1 will be infinitely large at first - if we do not want to take refuge in too few appropriate ideas. If δ is a finite size for infinitely large areas, then *X* and *Z* will generally be completely indeterminate, as long as no specific prerequisite is made about the way from finite values R_1 to the infinitely large ones. So, both quantities can just as well become infinite as remain finite. In other words, *X* and *Z* become completely undetermined and can become infinite if the mass density δ is finite within infinitely large parts of space. You can specify an infinite number of mass distributions, in which the acceleration *X*, i.e. also the velocity and also the strain, becomes infinitely large within finite or infinitely large distances. At the same time, consideration of the expressions (1) shows that under the given assumptions infinitely distant parts of space determine the motion and, as a result of the strain, also the nature of matter at a certain point. But since the mass distribution of infinitely distant parts of space is unfathomable for us, the mechanical states of matter in every point would also be unfathomable.

In order to clearly overlook these conditions, it is recommended to look at simple examples. Let's assume in a naturally completely arbitrary way first of all that the space is continuously filled with mass of the homogeneous density δ and further that we have to imagine the space as a sphere with an infinitely large radius. In mechanical terms, this would not eliminate the uncertainty of the situation, we also need to be able to specify the distance *r* of the attracted point from the center of the sphere. Then the point is accelerated towards the center of the sphere, which is proportional to $r\delta$. So, this acceleration has all values from zero to infinity in space. The distortion, on the other hand, remains finite everywhere and is proportional to δ. Secondly, we imagine a cone of any arbitrary, but very small opening ω. The same is filled with mass of the density δ, where δ may be only a function of the distance *r* from the tip *O* of the cone. If we call *a* the distance of the attracted point from *O* and γ the angle that this direction makes with the cone axis, the components of attraction in the direction of the axis (*X*) and perpendicular to it (*Y*) will be represented by the reduction of *a* and ω with any approximation by the formulas

$$X = \omega \int_{R_0}^{R_1} \delta \cdot \mathrm{d}r \left(1 + \frac{2a}{r} \cos\gamma\right)$$

$$Y = -\omega \int_{R_0}^{R_1} \delta \cdot \mathrm{d}r \frac{a}{r} \sin\gamma$$

where R_0 and R_1 are the boundaries of the truncated cone filled with mass. So here the attraction becomes infinite when it is R_1; but also the strain:

$$Z = 2\omega \int_{R_0}^{R_1} \delta \frac{\mathrm{d}r}{r}$$

in the direction of the axe becomes unlimited. If we think about the second part of the double cone, too, which is occupied by mass in exactly the same way, then for $a = 0$, *X*, for example, is at first quite indefinite, since it assumes the form ∞ - ∞ and we can give this expression any value we like by letting the extension of one cone into infinity depend on that of the other in a suitable way. Assuming that both cones are always the same size, the component $X = 0$, the attracted point is then in a kind of infinitely unstable equilibrium. But the distortion becomes twice as large as in the earlier case, i.e. for $R_1 = \infty$ also infinitely large and the matter could not exist at all near the cone tip.

From such simple and obvious examples, it is clear in any case that quite possible and imaginable assumptions lead to quite impossible or unthinkable consequences. However, one can hardly consider such occurrences to be admissible in the case of a generally valid law and must therefore conclude that Newton's law, applied to the immeasurably extended universe, leads to insurmountable difficulties and insoluble contradictions if one considers the matter scattered in the universe to be infinitely large.

It will therefore be necessary to make a choice between the two assumptions: 1) the total mass of the universe is immeasurably large, then Newton's law cannot be regarded as a mathematically strict expression for the prevailing gravitational forces, 2) Newton's law is absolutely accurate, then the total matter of the universe must be finite, or more precisely, infinitely large parts of space must not be filled with mass of finite density. As is well known, the question of whether the total matter is finite or infinitely large is answered in various ways, and I will certainly not claim to be able to reach a decision on this much-discussed question if I express my view that an absolutely empty space or a space filled with infinitely thin matter is not conceivable at all. However, the present question can also be viewed from a different angle. One may look at it as one likes, but it will always be difficult to make an evaluation of the basic mechanics of heaven dependent on its answer, and from the scientific point of view, the view that is completely independent of meta-physical speculations will undoubtedly be considered more appropriate and therefore more correct.

Now Newton's law is still a purely empirical formula, the accuracy of which would be a new and unsupported hypothesis if it were assumed to be absolute. Therefore, I do not think that it is doubtful that we are acting correctly if we do not recognize the absolute precision of Newton's law, but rather if we assume that we should receive such supplementary elements that the difficulties discussed will disappear of their own accord, while on the other hand, of course, the 'facts' observed in our planetary system will be satisfied. Of course, the necessary supplementary elements are not determined by these points of view and there are an infinite number of permissible assumptions. More to mention an example than to show a result of deeper insights, a suitable assumption shall be mentioned. The view that gravity is a suddenly affecting remote force, at the moment can no longer be maintained. However, if one assumes a medium that mediates the attraction, one will have to admit the possibility of the necessity of a correction coming from this source. This correction is still completely unknown for the time being. But it will not be considered absurd to consider, without prejudice to the expansion of our knowledge, one of the many existing analogies of attraction with other agents, namely with the light, all alone, for example, only. One would then have to think of a kind of absorption that gravity experiences in space. Whether this absorption occurs solely through the mediating medium itself, e.g. as a result of imperfect elasticity or the like, or whether it also occurs through masses in between, is better left undiscussed for the time being. The latter assumption, although somewhat unusual, cannot be rejected outright. According to this assumption, the attraction which two masses exert on each other would have to be influenced by the interposition of a third mass, and indeed reduced. So, for example, the attraction of the sun to the moon during a total lunar eclipse would have to be smaller than it otherwise would be. Whether such an influence, which can of course only be very small, can be proven or not, cannot be claimed without very detailed investigations.

The simplest formula that takes absorption into account is obtained by applying Newton's law $\frac{\kappa^2 mm'}{r^2}$ the factor $e^{-\lambda r}$ adds where *e* is the base of the natural logarithmic system. λ becomes not a constant attraction *A* is thus expressed by

$$A = \kappa^2 mm' \cdot \frac{e^{-\lambda r}}{r^2} \qquad (2)$$

It is obvious that λ can always be chosen so small that within our planetary system Newton's law emerges with any approximation. On the other hand, the above-mentioned difficulties have completely disappeared, because the integrals

$$\int_{R_0}^{R_1} \delta e^{-\lambda r} r \, \mathrm{d}r, \quad \int_{R_0}^{R_1} \delta \cdot e^{-\lambda r} \mathrm{d}r, \quad \int_{R_0}^{R_1} \delta \frac{e^{-\lambda r}}{r} \mathrm{d}r$$

have finite values and the expressions under the integral signs become infinitely small for $r = \infty$, so that the state and the motion of matter is no longer mainly determined by infinitely distant parts of space.

With not too small values of λ but gives (2) for the planetary movements reason for deviations from the Kepler's movement, which can become noticeable. The Keplerian motion appears disturbed by a force *R* acting in the radius vector, which can be set for a small λ

$$R = +\mu \frac{\lambda}{r} \qquad (3)$$

if for short cut $\mu = \kappa^2(1 + m)$ law. If one introduces into the known expressions for the variation of the elliptical elements (*e* eccentricity, *a* large half axis, χ length of the perihelion) the true anomaly *v* instead of time, then one has for any disturbing force *R*

$$\frac{\mathrm{d}e}{\mathrm{d}v} = \frac{Rr^2 \sin v}{\mu}, \qquad \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}v} = \frac{2eRr^2 \sin v}{\mu(1 - e^2)}, \qquad e\frac{\mathrm{d}\chi}{\mathrm{d}V} = -\frac{Rr^2 \cos v}{\mu}$$

The change in the average length may be omitted here. One sees now at first sight that with regard to (3) secular members of the first order can only be created in χ and since the periodic perturbations are quite insignificant for such small λ as are considered here, one has only the change of the perihelion length according to the formula

$$e\frac{\mathrm{d}\chi}{\mathrm{d}v} = -\lambda\, r \cos v = -\frac{\lambda a(1 - e^2) \cos v}{1 + e \cos v}$$

to be considered. The integration of this equation would be very easy to write down. For the purpose of the intended rough calculation, it is sufficient to take only the first power of *e* and then find

$$e(\chi - \chi_0) = \frac{\lambda a e}{2} v - \lambda\, a \sin v + \frac{\lambda a}{4} e \sin 2v$$

and the secular part of it is $\frac{\lambda a e}{2} v$ if *n is* also called the mean motion, then if the eccentricity is neglected $v = nt$ and one has thus for the saecular part $\Delta\chi$ of the change of the perihelion length during the time Δt:

$$\Delta\chi = +\frac{\lambda\, a\, n}{2} \Delta t$$

As is well known, such a forward movement of the perihelion of Mercury has been observed. According to Leverrier the other elements remain unchanged according to Newton's law. So this fact would correspond completely to the conclusions of formula (2). Meanwhile, the Leverrier's result might not yet be completely without contradiction and a note of Mr. Newcomb, published a few days ago,[2] promises proof that other empirical elements in the planetary system are also necessary. However, if we retain Leverrier's result for the time being $\Delta\chi = 40''$ to be set in the century. From this follows $\lambda = 0.00000038$. With this, however, now also secular movements of the perihelion lengths of the remaining planets result, namely for Venus, Earth, Mars, Jupiter, Saturn, Neptune respectively 29", 24", 20", 11", 8", 5" in the century, while for the moon only 0.9" follows. When asking whether such amounts should not have been shown in the observations, one should not forget that in the heliocentric planetary orbits the perihelion motion occurs only multiplied by quantities of the order of eccentricity and the mean motion changed by a constant

[2] Astronomical journal No. 327.

quantity is determined directly from the observations. From this it follows that an exact observation of the motion of Mars in this direction will give the easiest information about whether similar amounts as the above are permissible or not. But it is likely that Mr. Newcomb's detailed and highly detailed investigations of the movements of the inner planets, carried out with great resources, will soon bring about a decision on this and similar questions, so it would be useless to discuss this subject in detail. Also, the expected results only touch very superficially on the subject of this paper, because for the latter it is only an interesting coincidence that the formula (2), established without deeper justification, formally explains the movement of the perigelion of Mercury. If the anomaly in Mercury's motion could not be explained by an alteration of the law of attraction, this would only mean, in the sense of formula (2), that the size λ is much smaller than calculated above on the basis of Leverrier's view.

Recently, Mr.[3] A. Hall, with the intention of explaining the perihelion movement of Mercury, proposed another modification of Newton's law. He assumes that instead of r^2 in the terms of this Act $r^{\alpha+2}$ is to be set and finds

$$a = 0.00000016$$

This assumption also involves secular perihelion movements with the other planets, namely with Venus: 17", Earth 10" and with the moon 139". However, this interesting modification of Newton's law is not suitable for solving the above-mentioned difficulties. For in place of the integrals with respect to *r* in the three expressions (1), the integrals now take the place of the integrals for Mr. Hall's formula:

$$\int_{R_0}^{R_1} \delta\, r^{1-\alpha} dr\,; \qquad \int_{R_0}^{R_1} \delta \frac{dr}{r^{\alpha}}\,; \qquad \int_{R_0}^{R_1} \delta \frac{dr}{r^{1+\alpha}}$$

And the bland first become, as with Newton's law, infinitely large for infinitely large R_1.

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[3] Astronomical Journal No. 319.